D. Raji

Ahmadu Bello University, Zaria
Department of Mathematics
First Semester Examination 2023/2024 Session

MATH 309: Analytical Dynamics
Instruction: Answer Any Four Questions. Time: 2 hours

1. Obtain the radial and transverse components of velocity and acceleration of a particle moving in a plane curve.

2. (a) A point moves in a plane curve so that its tangential and normal accelerations are equal and the tangent rotates with a constant angular velocity. Find the path.
(b) Two equal uniform rods AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. If 2θ be the angle between them, establish a relation between a, b and θ.

3. State and prove the conservation theorem for the angular momentum of a system of particles.

4. Establish Euler's equations of motion for a rigid body.

5. State and prove Hamilton's Principle.

6. Obtain equations of motion for:
(a) a simple pendulum using Hamilton's Principle
(b) a planet orbiting round the Sun using Lagrange's equations.

MATH309: ANALYTICAL DYNAMICS 2023/2024

1. Let a particle moving in a plane curve be $\vec{r}(t)$ and the position at that time t be $p(R, \theta)$ then its position vector can be expressed as $\vec{r}(t) = R\hat{e}_R$ where $\hat{e}_R$ is a unit tangent to the R co-ordinate line.

The velocity and acceleration are $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{f} = \frac{d\vec{v}}{dt}$ respectively.

Thus, $\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\hat{e}_R + R\frac{d}{dt}(\hat{e}_R)$

But $\hat{e}_R = \cos\theta\hat{i} + \sin\theta\hat{j} \Rightarrow \frac{d}{dt}(\hat{e}_R) = \dot{\theta}(-\sin\theta)\hat{i} + \dot{\theta}\cos\theta\hat{j} = \dot{\theta}\hat{e}_\theta$

$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$ where $\hat{e}_\theta$ is a unit tangent to θ-coordinate.

Thus, $\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$

where the radial component of velocity $= \dot{R}$ and the transverse component is $R\dot{\theta}$ in the sence that both R and θ are increasing and the unit vector $\hat{e}_\theta$ and $\hat{e}_R$ are orthogonal.

Also,
$$\vec{f} = \frac{d\vec{v}}{dt} = \ddot{R}\hat{e}_R + \dot{R}\frac{d}{dt}(\hat{e}_R) + \dot{R}\dot{\theta}\hat{e}_\theta + R\ddot{\theta}\hat{e}_\theta + R\dot{\theta}\frac{d}{dt}(\hat{e}_\theta)$$
$$= \ddot{R}\hat{e}_R + \dot{R}\dot{\theta}\hat{e}_\theta + \dot{R}\dot{\theta}\hat{e}_\theta + R\ddot{\theta}\hat{e}_\theta + R\dot{\theta}(-\dot{\theta}\hat{e}_R)$$

Because $\frac{d}{dt}(\hat{e}_\theta) = \frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j}) = -\dot{\theta}\cos\theta\hat{i} - \dot{\theta}\sin\theta\hat{j} = -\dot{\theta}\hat{e}_R$

$$\Rightarrow \vec{f} = \ddot{R}\hat{e}_R + 2\dot{R}\dot{\theta}\hat{e}_\theta + R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_R = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$$

Hence, the radial component of acceleration $= \ddot{R} - R\dot{\theta}^2$ and the transverse acceleration $= 2\dot{R}\dot{\theta} + R\ddot{\theta}$ respectively.

2a) Tangential acceleration is given by $\ddot{S}$ and normal acceleration is $\frac{\dot{S}^2}{\rho}$ where $\rho = \frac{ds}{d\psi}$ $\Rightarrow \ddot{S} = \frac{\dot{S}^2}{\rho} \Rightarrow \frac{d\dot{s}}{dt} = \dot{s}^2\frac{d\psi}{ds} \Rightarrow \frac{dv}{ds}\cdot\frac{ds}{dt} = v^2\frac{d\psi}{ds}$

$\Rightarrow v\frac{dv}{ds} = v^2\frac{d\psi}{ds}$ where $\frac{ds}{dt} = v \Rightarrow \frac{1}{v}dv = d\psi$

Integrating bothsides gives $\int\frac{dv}{v} = \int d\psi \Rightarrow \ln v = \psi + \ln c$

$\Rightarrow \ln v - \ln c = \psi \Rightarrow \ln(v/c) = \psi \Rightarrow v/c = e^\psi \Rightarrow v = ce^\psi$

$\Rightarrow \frac{ds}{dt} = ce^\psi \Rightarrow \frac{ds}{d\psi}\cdot\frac{d\psi}{dt} = ce^\psi$

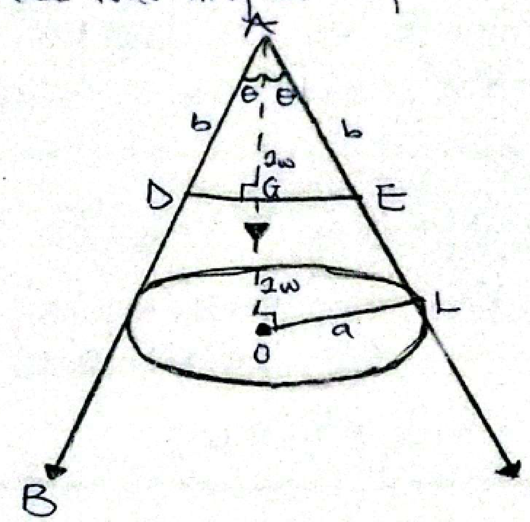
But $\frac{d\psi}{dt}$ = constant' say K $\Rightarrow Kds = ce^\psi d\psi$

Integrating bothsides

$\int Kds = \int ce^\psi d\psi \Rightarrow Ks = ce^\psi + D$

Which is the required equation of the path.

(2b) Let D and E be the middle points of AB and AC respectively. Then $AD = \frac{1}{2}AB = b$ and $AE = \frac{1}{2}AC = b$. G is the common gravity of the two rods AB and AC where the total weight 2w of the two rods may be suspended to act vertically downwards from $\triangle AGD$



$$AG = AD\cos\theta = b\cos\theta \quad \text{——— (1)}$$

from $\triangle AOL$,

$$\sin\theta = \frac{OL}{OA} = \frac{a}{OA}$$

But $OA = \frac{a}{\sin\theta} = a\,\text{cosec}\,\theta$

$$\therefore OG = OA - AG$$

$$= a\,\text{cosec}\,\theta - b\cos\theta \quad \text{——— (2)}$$

Let assume that the rods undergo a small symmetrical displacement about the vertical through the centre O such that the angle θ changes into $\theta + \delta\theta$. Then the equation of the virtual work $\Rightarrow -2w\,\delta(OG) = 0$ where the negative sign indicate that the distance is measured in the direction opposite to that force $\Rightarrow \delta(OG) = 0$ since $w \neq 0$

or $\delta(a\,\text{cosec}\,\theta - b\cos\theta) = 0$ or $(-a\,\text{cosec}\,\theta\cot\theta + b\sin\theta)\,d\theta = 0$

or $(-a\,\text{cosec}\,\theta\cot\theta + b\sin\theta) = 0$ since $\delta\theta \neq 0$

$$\therefore \cancel{-}\frac{a\cos\theta}{\sin^2\theta} = \cancel{-}b\sin\theta \Rightarrow \underline{a\cos\theta = b\sin^3\theta}$$

(3) The law of conservation of angular momentum of a system of particles states that if the sum of the external torques about a fixed point acting on a system of a particle is zero, then the angular momentum of the system about the fixed point is constant throughout the motion.

Proof:

Consider a mass-system S consisting of n-particles P_i each with a constant mass m_i and position vector $\vec{r_i}$ with respect to the origin O of an inertia frame. Then by Newton's second law, the equation of motion of the particle P_i can be written as;

$$\frac{d}{dt}(m_i\dot{\vec{r_i}}) = \vec{F_i} + \sum_{k=1}^{n}\vec{f}_{ik} \quad \text{——— (1)}$$

Where $\vec{F_i}$ is the total external forces acting on the particle P_i and $\vec{f}_{ik}$ is the internal force which the particle P_k exert on the particle P_i

The rate of change of the angular momentum, we take the vector product of eqn (1) with the position vector $\vec{r_i}$ to get;

$$\vec{r_i} \times \frac{d}{dt}(m_i\dot{\vec{r_i}}) = \vec{r_i} \times \vec{f_i} + \sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}} \quad \text{—— (2)}$$

The angular momentum of the particle P_i is

$$\vec{H_i} = \vec{r_i} \times m_i\dot{\vec{r_i}}$$

Now, $\frac{d\vec{H_i}}{dt} = \frac{d}{dt}(\vec{r_i} \times m_i\dot{\vec{r_i}}) = \vec{r_i} \times \frac{d}{dt}(m_i\dot{\vec{r_i}}) + \dot{\vec{r_i}} \times m_i\dot{\vec{r_i}}$

$$= \vec{r_i} \times \frac{d}{dt}(m_i\dot{\vec{r_i}}) \quad \text{—— (3)} \quad \text{since } (\dot{\vec{r_i}} \times \dot{\vec{r_i}}) = 0$$

from (2) and (3), we have;

$$\frac{d\vec{H_i}}{dt} = \vec{r_i} \times \vec{f_i} + \sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}}$$

Summing over all particles, we have

$$\frac{d\vec{H}}{dt} = \sum_{k=1}^{n} \frac{d\vec{H_i}}{dt} = \sum_{i=1}^{n} \vec{r_i} \times \vec{f_i} + \sum_{i=1}^{n}\sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}} \quad \text{—— (4)}$$

Since the internal forces are central and the torques due to the interacting particles are positively directed by equal in magnitude, then

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}} = 0$$

Therefore, (4) becomes $\frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \vec{r_i} \times \vec{f_i}$ —— (5)

If the sums of the external torques about a fixed point acting on the system is zero i.e $\sum_{i=1}^{n} \vec{r_i} \times \vec{f_i} = 0$ then, eqn (5) reduces to

$$\frac{d\vec{H}}{dt} = 0.$$

$\square$.

(4.) Euler's equations of motion for a rigid body.

Consider a rigid body which turns about a point O fixed in both body and space. Let I_1, I_2, I_3 stands for the principle moments of inertia I and $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity $\vec{\omega}$ at any time. If $\vec{i}, \vec{j}, \vec{k}$ stipulate the unit vectors along these axes, that is the principle axes 1, 2, 3 which are essentially mutually orthogonal and fixed in the body, then the moments of momentum (angular momentum) about O is instantaneously

$$\vec{L} = I_1\omega_1\vec{i} + I_2\omega_2\vec{j} + I_3\omega_3\vec{k} \quad \text{—— (1)}$$

The motion of a rigid body with one point fixed will take place under the action of a torque $\vec{N}$ in such a way that its total angular momentum varies at the rate equal to $\vec{N}$ —— (2)

$$\frac{d\vec{L}}{dt} = \vec{N}$$

The time-derivative refers to the space axes, the equation holds only in an inertia system. In a coordinate system rotating between the two time derivatives,

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times \quad \text{--- (3)}$$

Eqn (2) in terms of body axes is therefore;

$$\left(\frac{dL}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N} \quad \text{--- (4)}$$

In the case of (1), we remember that the principal moments of inertia and the body base vectors $\vec{i}, \vec{j}, \vec{k}$ are constant in time with respect to the time derivative of $\vec{L}$ i.e $\frac{d\vec{L}}{dt}$ in rotating system is;

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = I_1\dot{\omega}_1\vec{i} + I_2\dot{\omega}_2\vec{j} + I_3\dot{\omega}_3\vec{k} \quad \text{--- (5)}$$

$$\omega\times\vec{L} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= (I_3-I_2)\omega_2\omega_3\vec{i} - (I_3-I_1)\omega_3\omega_1\vec{j} + (I_2-I_1)\omega_1\omega_2\vec{k} \quad \text{--- (6)}$$

Writing $\vec{N} = N_1\vec{i} + N_2\vec{j} + N_3\vec{k}$ and using (5) and (6) in (4) and comparing the coefficient of $\vec{i}, \vec{j}, \vec{k}$, we have;

$$\left.\begin{aligned} N_1 &= I_1\dot{\omega}_1 - (I_2-I_3)\omega_2\omega_3 \\ N_2 &= I_2\dot{\omega}_2 - (I_3-I_1)\omega_3\omega_1 \\ N_3 &= I_3\dot{\omega}_3 - (I_1-I_2)\omega_1\omega_2 \end{aligned}\right\} \quad \text{--- (7)}$$

Eqn (7) are known as "Euler's dynamical equations" for the motion of a rigid body with one point fixed. For principal axes, the resultant moment of the external forces is zero i.e eqn (7) becomes;

$$\left.\begin{aligned} I_1\dot{\omega}_1 - (I_2-I_3)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 - (I_3-I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 - (I_1-I_2)\omega_1\omega_2 &= 0 \end{aligned}\right\} \quad \text{--- (8)}$$

Multiplying (8) by $\omega_1, \omega_2, \omega_3$ and adding gives;

$$I_1\dot{\omega}_1\omega_1 + I_2\dot{\omega}_2\omega_2 + I_3\dot{\omega}_3\omega_3 = 0$$

On integrating, $I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \text{Constant}$ --- (9)

Since the LHS of (9) is $2T$ where T is the kinetic energy. Hence (9) express the constancy of the K.E during the motion. Since ~~the~~ no work is being done ~~on~~ the body, there being no external forces, another integral of (8) can be obtained by multiplying them by $I_1\omega_1, I_2\omega_2, I_3\omega_3$ and adding gives;

$$I_1^2 w_1^2 + I_2^2 w_2^2 + I_3^2 w_3^2 = \text{Constant} \quad \text{——} (10)$$

Eqn (10) expresses the constancy of the magnitude of the angular momentum L during the motion.

5) Out of all possible paths along which a dynamical system may move from one point to another in the configuration space within a specified time interval, the actual path is that for which the time integral of the lagrangian of the system is an extremum (minimum or maximum). The Hamilton's principle in terms of calculus of variation is expressed as; $\partial \int_{t_1}^{t_2} L\, dt = 0$ with variation zero at time $t = t_1$ and $t = t_2$.

Proof: Consider a holonomic conservative dynamical system whose configuration space can be described by n generalized co-ordinates $q_j (j = 1, 2, 3 \cdots n)$ at any instant. Then the Lagrangian of the system $L = T - V$ where T is a function of generalized coordinate and their time derivative & possible time t and V is the function of generalized co-ordinates only. The Lagrangian can be expressed as;

$$L = L(q_1, q_2, q_3 \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots \dot{q}_n, t) = L(q_j, \dot{q}_j, t)$$

The variation when holding time fixed can be written as;

$$\partial L = \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} \partial q_j + \frac{\partial L}{\partial \dot{q}_j} \partial \dot{q}_j \right]$$

Now, $\partial \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \partial L\, dt = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} \partial q_j + \frac{\partial L}{\partial \dot{q}_j} \partial \dot{q}_j \right) dt$

$$= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_j} \partial q_j + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial}{\partial t} (\partial q_j) \right] dt$$

$$= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} \right) \partial q_j + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial}{\partial t} (\partial q_j) \Big] dt$$

$$= \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \partial q_j \right) dt \Big] = 0$$

$$= \int_{t_1}^{t_2} \partial \left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \cdot \partial q_j \right) = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \partial q_j \Big|_{t_1}^{t_2}$$

$$= \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \partial q_j \Big|_{\text{at } t = t_2} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \partial q_j \Big|_{\text{at } t = t_1} = 0 - 0 = 0$$

Thus, $\partial \int_{t_1}^{t_2} L\, dt = 0$

Since $\partial q_j = 0$ at $t = t_1, t_2$.

6a) By Hamilton's principle

$$\partial\int_{t_1}^{t_2} L\,dt = 0 \Rightarrow \int_{t_1}^{t_2} \partial L\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\left[\tfrac{1}{2}ml^2\cdot 2\dot\theta\,\partial\dot\theta - mgl(\theta + \sin\theta)\right]dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\left[ml^2\dot\theta\frac{d}{dt}(\partial\theta) - mgl\sin\theta\,\partial\theta\right]dt = 0$$

$$= ml^2\dot\theta\,\partial\theta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} ml^2\ddot\theta\,\partial\theta\,dt$$

$$= -\int_{t_1}^{t_2} mgl\sin\theta\,\partial\theta\,dt = 0$$

$\because\ \partial\theta = 0$ at $t = t_1$ and $t = t_2$

$$0 - \int_{t_1}^{t_2}(ml^2\ddot\theta + mgl\sin\theta)\,\partial\theta\,dt = 0$$

$$\Rightarrow ml^2\ddot\theta + mgl\sin\theta = 0$$

$$\Rightarrow \ddot\theta + \mu\theta = 0$$

Since θ is very small, $\mu = g/l$.

Scanned with CamScanner

(6b) Equation of motion for a planet orbiting round the sun using Lagrange's equation.

Consider a planet P of mass m orbiting round the sun S (fixed) Under the inverse square law of attraction μm. Let (r, θ) be the polar coordinate of P w.r.t S at any time t. Then these are generalised coordinate of the system.

Kinetic energy $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and

potential energy $V = -\int_{\infty}^{r} F\,dr = -\int_{\infty}^{r} -\frac{k}{r^2}\,dr$ where $k = \mu m$

and the negative sign denotes attraction towards the sun.

$$\therefore V = -\int_{\infty}^{r} -kr^{-2}\,dr = -\left[\frac{-kr^{-1}}{-1}\right]_{\infty}^{r} = \frac{-k}{r}\Big|_{\infty}^{r} = -k\left(\frac{1}{r}\right)\Big|_{\infty}^{r} = \frac{-k}{r}$$

By Lagrangian,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

Lagrange's equation of motion, we have;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{——— radian motion}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{——— Angular motion.}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{r}} = m\dot{r} \cdot \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r^2}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

By substituting into radian & angular motion, we have;

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0 \Rightarrow m(\ddot{r} - r\dot{\theta}^2) + \frac{k}{r^2} = 0$$

and

$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = 0 \Rightarrow mr^2\ddot{\theta} = 0$$

Which is the required equation of motion for a planet orbiting the sun.